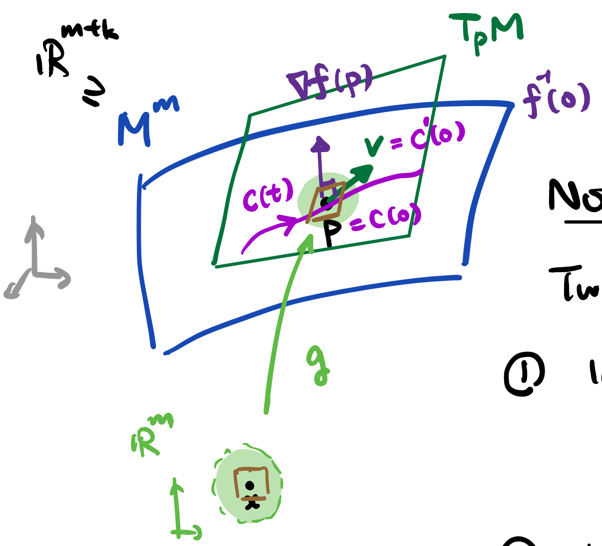
§ Tangent Bundle

Motivation: $p \in M^m \subseteq \mathbb{R}^{m+k}$ submanifold



$$T_pM := \{ v \in \mathbb{R}^{m+k} \mid \exists \text{ smooth } c : (-\varepsilon, \varepsilon) \to M \text{ s.t. } c(0) = p,\ c'(0) = v \}$$

Note: T_pM is an m-dim'l subspace in $\mathbb{R}^{m+k}$

Two ways to describe this subspace:

① locally, $M = f^{-1}(0)$ for some $f : \mathcal{U} \subseteq \mathbb{R}^{m+k} \to \mathbb{R}^k$

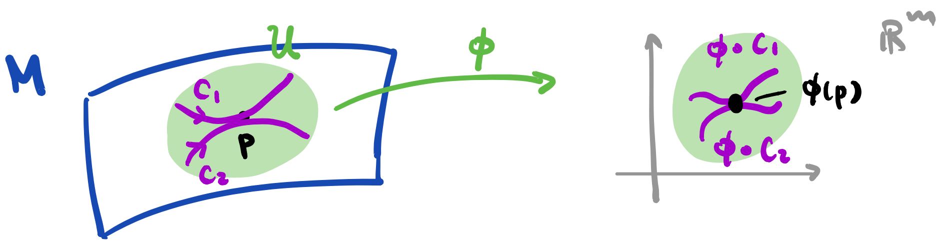
$\Rightarrow T_pM = \ker(df_p)$. dim: $(m+k) - k = m$

② locally, parametrization $g : W \subseteq \mathbb{R}^m \to M \subseteq \mathbb{R}^{m+k}$

$\Rightarrow T_pM = dg_x(\mathbb{R}^m)$ dim $= m$

Q: How to define T_pM in the setting of abstract manifolds?

Def^{n}: Let $p \in M$. Given curves $c_i : I_i \to M$, $i = 1, 2$, where $I_1, I_2 \subseteq \mathbb{R}$ open intervals containing 0 s.t. $c_1(0) = p = c_2(0)$.



We say $c_1 \sim c_2$ iff $\exists$ chart $(\mathcal{U}, \phi)$ around p s.t.

Ex: This is an equivalence relation.

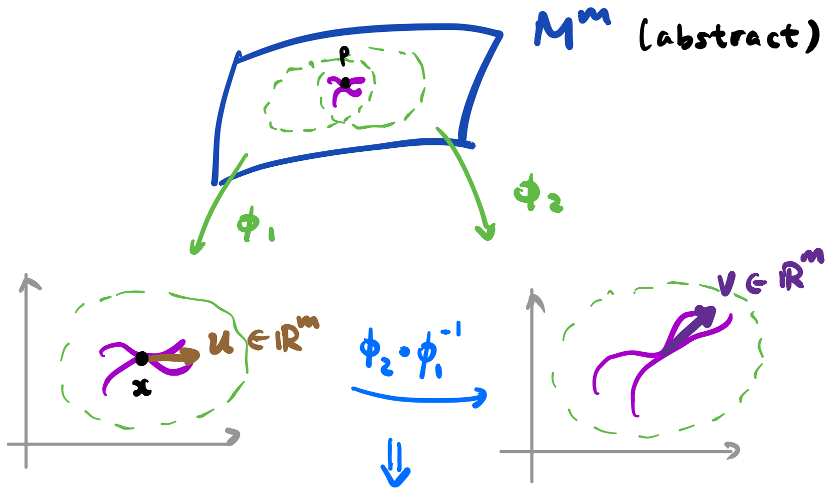
$$(\phi \circ c_1)'(0) = (\phi \circ c_2)'(0) \quad \text{inside } \mathbb{R}^m.$$

$$T_pM := \{ [c] \mid c : I \to M \text{ curve s.t. } c(0) = p \}$$

($0 \in I$)

<u>Remark</u>: • T_pM m-dim'l (abstract) vector space.

* • The relation $\sim$ above is "chart-independent".



$$\Downarrow$$

$$v = d_x(\phi_2 \circ \phi_1^{-1})(u)$$

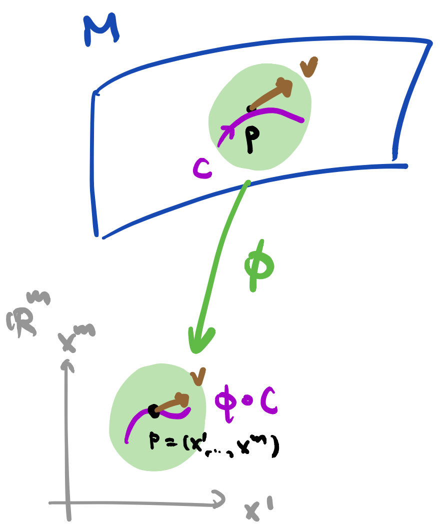
<u>Def^{n}</u>: $TM := \coprod_{p \in M} T_pM = \{(p,v) : p \in M,\ v \in T_pM\}$.

Tangent Bundle of M

disjoint union

<u>Thm</u>: TM is a smooth manifold (of dim $= 2 \cdot \dim M$)

"<u>Why?</u>" Describe the local charts for TM.



$$(c(0), c'(0)) = (\underset{\substack{\cup\\ M}}{p}, \underset{\substack{\cup\\ T_pM}}{v}) \in TM$$

$\downarrow$ local coord

$$\Big(\underbrace{\phi \circ c(0)}_{\substack{(x^1, \dots, x^m) \\ \cap \\ \mathbb{R}^m}}, \underbrace{(\phi \circ c)'(0)}_{\substack{\cap \\ \mathbb{R}^m}}\Big) \in \mathbb{R}^{2m}$$

$\phi_2 \circ \phi_1^{-1}$ smooth

<u>Transition maps</u>: $(\phi_1 \circ c(0), (\phi_1 \circ c)'(0)) \longleftrightarrow (\phi_2 \circ c(0), (\phi_2 \circ c)'(0))$

$d(\phi_2 \circ \phi_1^{-1})_{\phi_1(\cdot)}$ smooth

§ Vector Bundles

Def^{n}: A vector bundle (of rank n) consists of a map

$$\underset{\text{(total space)}}{\pi : E} \longrightarrow \underset{\text{(base space)}}{B}$$

Notation:

$$\begin{array}{c} \mathbb{R}^n \to E \\ \downarrow \pi \\ B \end{array}$$

s.t. (1) E, B smooth maps, π smooth, onto

(2) $\exists$ open cover $\{U_i\}_{i \in I}$ of B and

$\exists$ diffeo $h_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$

local trivializations s.t. $h_i(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

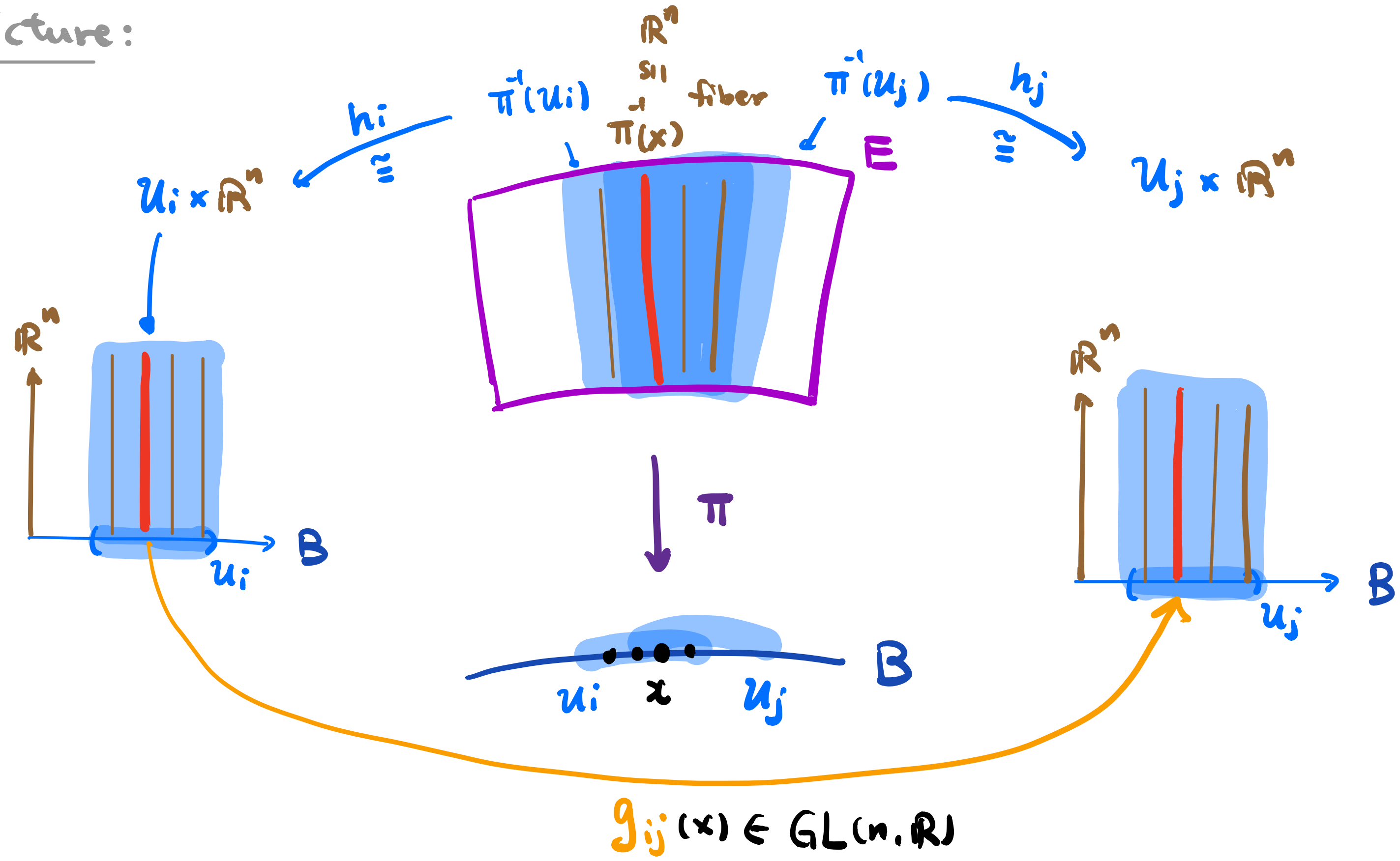
(3) The "transition maps" $h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \xrightarrow{\cong} (U_i \cap U_j) \times \mathbb{R}^n$

are diffeomorphisms of the form:

$$h_i \circ h_j^{-1}(x, v) = (x, g_{ij}(x) \cdot v)$$

where $g_{ij} : U_i \cap U_j \longrightarrow GL(n, \mathbb{R})$ smooth (in x).

Picture:



Examples: (i) $M \times \mathbb{R}^n$ "trivial bundle".

(ii) TM is a rank m vector bundle, where $m = \dim M$.

$$\begin{array}{ccc} \mathbb{R}^m \to & TM & \ni (p, \overset{[c]}{v}) \\ & \pi \downarrow & \downarrow \\ & M & \ni p \end{array}$$

$\{(U_i, \phi_i)\}$ chart on M

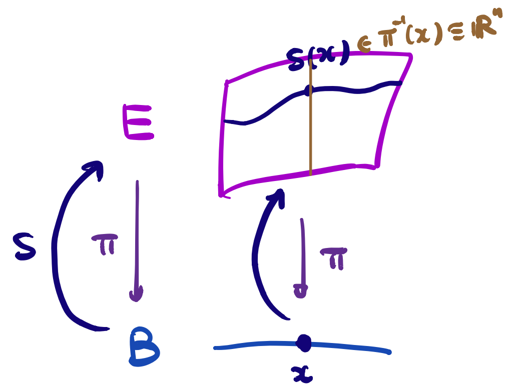
local trivialization

$$h_i : \pi^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{R}^n$$
$$(p, v) \longmapsto (\phi_i(p), (\phi_i \circ c)'(0))$$

$$g_{ij}^{(x)} = d(\phi_j \circ \phi_i^{-1})_{\phi_i(x)} \in GL(n, \mathbb{R})$$

Def$^{\underline{n}}$: A vector bundle (of rank n) $\pi : E \to B$ is trivial if $\exists$ diffeo $h : E \xrightarrow{\cong} B \times \mathbb{R}^n$ s.t. it is fiberwise linear isomorphism, i.e. $h : \pi^{-1}(x) \xrightarrow{\cong} \{x\} \times \mathbb{R}^n$.

Def$^{\underline{n}}$: A smooth map $S : B \to E$ is called a section of the vector bundle $\pi : E \to B$ if $\pi \circ S = id_B$.



E.g.: $E = B \times \mathbb{R}^n$

A section $S : B \to \mathbb{R}^n$ is a vector-valued function

§ Vector Fields on manifolds

Let M^m be a smooth m-manifold, tangent bundle TM.

Def$^{\underline{n}}$: A vector field on M is just a section $X : M \to TM$ of the tangent bundle TM.

Notation: $\Gamma(TM) := \{\text{sections of } TM\}$ (∞-dim'l vector space)

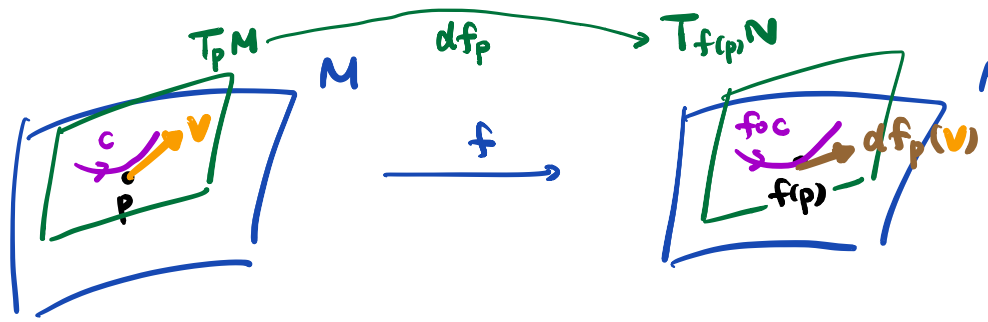
Def$^{\underline{n}}$: (Pushforward of tangent vectors)

Given smooth map $f: M \to N$, and $p \in M$,

then $\exists$ a linear map, differential of f at p.

$$df_p : T_pM \longrightarrow T_{f(p)}N$$

defined by $df_p(c'(0)) = (f \circ c)'(0)$ where $c: I \to M$, $c(0) = p$



Note: df_p is indep. of the choice of c representing $v \in T_pM$

Chain Rule: $\boxed{d(g \circ f)_p = dg_{f(p)} \circ df_p}$

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & N \arrow[r, "g"] & P \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} T_pM \arrow[r, "df_p"] \arrow[rr, bend right, "dg_{f(p)} \circ df_p"'] & T_{f(p)}N \arrow[r, "dg_{f(p)}"] & T_{g(f(p))}P \end{tikzcd}$$

Digression: Vector Fields on S^n.

$S^n \subseteq \mathbb{R}^{n+1}$ (unit sphere centered at 0)

$$TS^n = \{ (p, v) \in S^n \times \mathbb{R}^{n+1} \mid \langle p, v \rangle_{\mathbb{R}^{n+1}} = 0 \}$$

$$T(TS^n) = \{ X : S^n \to \mathbb{R}^{n+1} \text{ smooth} \mid \langle p, X(p) \rangle = 0 \ \forall p \in S^n \}$$

"Thm": TM trivial $\iff \exists$ m linearly indep. vector fields on M.

Def$^{\underline{n}}$: M is parallelizable if TM is trivial.

<u>Hard Thm 1</u> : All closed orientable 3-manifolds are parallelizable.

<u>Hard Thm 2</u> : S^n is parallelizable <u>iff</u> $n = 1, 3$ and 7

<u>Thm</u>: (Higher dim'l "Hairy Ball Theorem") $S^3 \times \mathbb{R}^3 \approx$

Any $X \in \Gamma(TS^n)$ must vanish somewhere when n is <u>even</u>

<u>Remarks</u> : • Thm $\Rightarrow$ TS^n is <u>NOT</u> trivial when n is even

• $n=2$ follows from Poincaré-Hopf Thm:

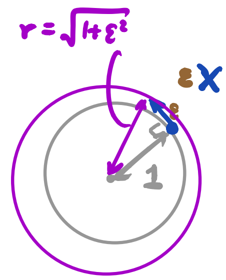
$$\sum_{\substack{p \in M \\ X(p)=0}} \text{index}\, X(p) = \chi(S^2) = 2 \neq 0$$

<u>Sketch of Proof</u> ($n \geq 4$, Milnor)

Suppose $\exists$ nowhere vanishing vector field X on S^n.

$\Downarrow$

WLOG, normalized to $\|X\| \equiv 1$.



Define $f: S^n(1) \xrightarrow{\cong} S^n(\sqrt{1+\varepsilon^2}^{\,=:r})$ <u>diffeo</u>.

$$x \longmapsto x + \varepsilon X(x)$$

$$d\mathrm{Vol}_{\mathbb{R}^{n+1}} = dx^0 \wedge \cdots \wedge dx^n = \frac{1}{n+1}\, d\omega$$

where $\omega := \sum_{i=0}^{n} (-1)^i x^i dx^0 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$ ($(n-1)$ form on $\mathbb{R}^{n+1}$)

$$\text{polynomial in } \varepsilon \approx \int_{S^n(1)} f^*\omega \overset{\text{change of var.}}{=} \int_{S^n(r)} \omega \overset{\text{Stokes'}}{=} \int_{B^{n+1}(r)} d\omega = \underbrace{(n+1)\,\mathrm{Vol}(B^{n+1}(r))}_{(n+1)\,\mathrm{Vol}(B^{n+1}(1))\, r^{n+1}}$$

$$r = \sqrt{1+\varepsilon^2}$$

$$= c \cdot (1+\varepsilon^2)^{\frac{n+1}{2}}$$

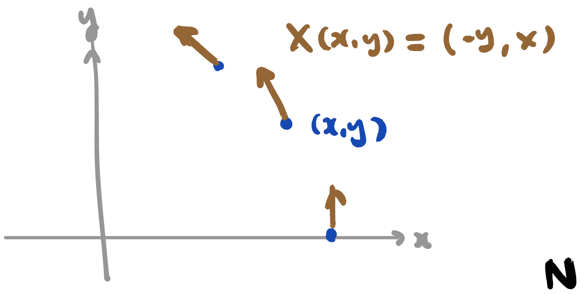
Contradiction when n even

$\blacksquare$

§ Vector Fields as "derivations"

$X \in \Gamma(TM)$ locally in coord $X(x_1,\dots,x_m) = \sum_{i=1}^{m} X^i(x_1,\dots,x_m) \frac{\partial}{\partial x^i}$

E.g.) In $\mathbb{R}^2$,



Write: $X(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Let $f(x,y) = xy$.

$$X(f) = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = -y^2 + x^2$$

Note: $X : f \mapsto X(f)$

IDEA: X acts on smooth functions $C^\infty(M)$ by directional derivative

Notation: $C^\infty(M) := \{ f: M \to \mathbb{R} \text{ smooth} \}$.

$\mathrm{Diff}(M) := \{ \varphi : M \to M \text{ diffeo.} \}$.

Given $X \in \Gamma(TM)$, $f \in C^\infty(M)$, $p \in M$,

$$X(f)(p) := \sum_{i=1}^{m} X^i(0) \frac{\partial f}{\partial x^i}\Big|_0$$

for any local coord. $x^1,\dots,x^m$ s.t. $p=0$.

Consider all points $p \in M$,

$$\begin{array}{ccccc} \Gamma(TM) \ni X : & C^\infty(M) & \longrightarrow & C^\infty(M) \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & f & \longmapsto & X(f) \end{array}$$

Prop: The map above is a derivation, i.e. $\forall\, a,b \in \mathbb{R}$, $f,g \in C^\infty(M)$,

(1) "Linearity": $X(af + bg) = a\,X(f) + b\,X(g)$

(2) "Liebniz Rule": $X(fg) = g \cdot X(f) + f \cdot X(g)$

FACT: $\left\{ \begin{matrix} \text{vector fields} \\ \text{on } M \end{matrix} \right\} \xleftrightarrow[\text{corr.}]{1-1} \left\{ \begin{matrix} \text{derivations} \\ \text{on } M \end{matrix} \right\}$

Def^{n}: (Lie bracket) Let $X, Y \in \Gamma(TM)$.

$$[X, Y] := XY - YX \quad \in \Gamma(TM).$$

i.e. $[X,Y](f) := X(Y(f)) - Y(X(f))$

Properties of $[\cdot,\cdot]$

(i) $[X,Y] = -[Y,X]$

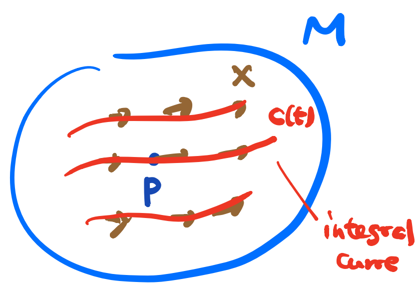
(ii) $[\cdot,\cdot]$ is $\mathbb{R}$-linear in each slot

(iii) (Jacobi identity) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Caution: $[\cdot,\cdot]$ is defined only using the smooth structure on M.

§ Flow and integral curves of vector fields

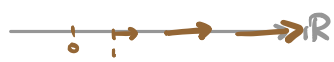
Let $X \in \Gamma(TM)$. Consider the following I.V.P.



$$\begin{cases} c_p'(t) = X(c_p(t)) & \forall t \in I \\ c_p(0) = p \end{cases}$$

O.D.E. $\Rightarrow$ $\exists$ unique sol^{n} $c_p(t) : \mathring{I}_p \to M$ that depends smoothly on the initial data $c(0) = p$

E.g. $X = x^2 \frac{\partial}{\partial x}$.



Thm: If $X \in \Gamma(TM)$ is compactly supported, then the maps

$$\begin{aligned} \phi_t : M &\longrightarrow M \\ p &\longmapsto c_p(t) \end{aligned}$$

is a diffeo. for each $t \in \mathbb{R}$.

Moreover, $\phi_t \circ \phi_s = \phi_{t+s} \quad \forall\, t, s \in \mathbb{R}$.

i.e. $\{\phi_t\}_{t \in \mathbb{R}} \subseteq \mathrm{Diff}(M)$ forms a 1-parameter group

called the flow generated by X.

Remarks: • If X not cptly supported, we can still define maps locally.

• Any $\phi \in \mathrm{Diff}(M)$ induces a pushforward map

$$\phi_* : \Gamma(TM) \to \Gamma(TM)$$

by the differential $d\phi_p : T_pM \to T_{\phi(p)}M$ at each $p \in M$.

Thm: Let $X, Y \in \Gamma(TM)$, cptly supported.

Suppose $\{\phi_t\}_{t \in \mathbb{R}}$ is the flow generated by Y.

Then.
$$[X, Y] = \frac{d}{dt}\Big|_{t=0} (\phi_t)_* X \quad \left(=: -\mathcal{L}_Y X\right)$$